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W) LINEAR NON-LINEAR WAVE PROPAGATION  
IONIZED MEDIA

Non-linear plasma waves in a uniform homogeneous,  
cold ionized medium

BY  
O. E. H. RYDBECK

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CHALMERS UNIVERSITY OF TECHNOLOGY

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GOTHENBURG S, SWEDEN



**DYNAMIC NON-LINEAR WAVE PROPAGATION  
IN IONIZED MEDIA**

**7. Non-linear plasma waves in a streaming  
homogeneous, cold ionized medium**

**by**

**O.E.H. Rydbeck**

**Gothenburg, 1962**

1. Summary.

The research reported in this communication deals with the properties of non-linear travelling and standing plasma waves in homogeneous, ionized cold streams.

For the travelling non-linear plasma waves it is possible to obtain direct and simple expressions (provided electronic overtaking does not occur), which relate the charge density to the true stream coordinate. This direct method is, unfortunately, not in general applicable to the practically important case of standing plasma waves - a symmetric space charge wave pair in the linear theory. There exists one exception, however, when the direct method can be applied to standing plasma waves, viz. the very special case of stream plasma resonance. In all other cases, provided there is no electronic overtaking, the indirect, but nevertheless exact, methods of Olving [1] will have to be used.

The non-linear maximum charge peaks, or charge bunches, are shown to be rather similar for the physically otherwise quite different travelling and standing non-linear waves. Both rapidly become very rich in harmonics as overtaking is approached. If such a stream runs through a multi-resonant system, for example the solar corona, an ionized anisotropic medium, or a series of cavities in a microwave device, high harmonics may be excited in the system. Whether a net harmonic radiation actually takes place or not from the interacting medium is a problem of its own, difficult and beyond the aims of the present investigation.

## 2. Introduction.

We assume that the ions and electrons of the infinitely wide, homogeneous and cold stream have a mean and equal density  $N_0$  and a drift velocity  $v_0$ , along the  $z$ -axis of the system. Furthermore, we neglect the oscillations of the ions and assume that the electrons make longitudinal oscillations only. This means that only electronic plasma oscillations are excited in the streaming medium.

We next introduce the following notations, viz.

$$\omega_p = \left( \frac{N_0 e^2}{m \epsilon_0} \right)^{1/2} = \text{the angular electronic plasma frequency,} \quad (1)$$

$$\rho = -Ne = -(N_0 + N_{\sim})e = \rho_0 + \rho_{\sim}, \quad (2)$$

where  $N_{\sim}$  is the differential electron density and  $\rho_{\sim}$  the differential - or ac - space charge density,

$$v = v_0 + v_{\sim}, \quad (3)$$

where  $v_{\sim}$  is the differential - or ac - velocity (in the  $z$ -direction),

$$z = z_0 + z_{\sim}, \quad (4)$$

where  $z_{\sim}$  is the displacement of the electron planes from the equilibrium position  $z_0 (= v_0 t$ , if the unperturbed electron plane in question left the  $z_0 = 0$  level at time  $t = 0$ ), and finally

$$E_z = E_z^0 + E_{\sim} = E_{\sim}, \text{ since } E_z^0 = 0, \quad (5)$$

where  $E_{\sim}$  is the axial ac electric field strength.

As long as  $z_{\sim}$  is a single valued function, i.e. when

there is no electronic overtaking, the electronic polarisation becomes

$$P_{\sim} = \rho_0 z_{\sim} = - N_0 e z_{\sim}. \quad (6)$$

The basic relation (6) holds for all non-linear oscillations, when  $z_{\sim}$  is single valued, which we assume to be the case in what follows.

Since we have an infinitely wide stream,

$$\epsilon_0 E_{\sim} + P_{\sim} = 0, \quad (7)$$

and

$$z_{\sim} = - \epsilon_0 E_{\sim} / \rho_0. \quad (8)$$

The equation of motion,

$$m \frac{d^2 z_{\sim}}{dt^2} = - e E_{\sim}, \quad (9)$$

therefore, by (8), immediately yields

$$\left( \frac{d^2}{dt^2} + \omega_p^2 \right) z_{\sim} = 0. \quad (10)$$

The electrons thus, even in the non-linear case, oscillate harmonically around their undisturbed position  $z_0 (= v_0 t)$  with the plasma frequency  $\omega_p/2\pi$ , a fact already pointed out and discussed by Olving [1].

By (4) rel. (10) can also be written

$$\left( \frac{d^2}{dz_0^2} + k_p^2 \right) z_{\sim} = 0, \quad (10a)$$

where

$$k_p^2 = \omega_p^2 / v_0^2 = (2\pi/\lambda_p)^2, \quad (11)$$

and  $\lambda_p$  is the plasma wavelength of the streaming medium.

The amplitude of the plasma oscillations obviously varies from plane to plane and can be regarded as a function of  $t - \frac{z_0}{v_0}$ , [1], which is a constant for any given electron (or plane of electrons).

Rel. (7) immediately yields

$$\epsilon_0 \frac{\partial E_{\sim}}{\partial t} + \frac{\partial P_{\sim}}{\partial t} = \epsilon_0 \frac{\partial E_{\sim}}{\partial t} + \rho_0 \frac{\partial z_{\sim}}{\partial t} = 0, \quad (12)$$

as it should be (the total axial ac current must be zero), i.e.

$$i_{\sim} = \frac{\partial P_{\sim}}{\partial t} = \rho_0 \frac{\partial z_{\sim}}{\partial t}, \quad (13)$$

where  $i_{\sim}$  is the (axial) ac convection current density.

Finally the equation of continuity,

$$\frac{\partial \rho_{\sim}}{\partial t} = - \frac{\partial i_{\sim}}{\partial z}, \quad (14)$$

by (13) requires that

$$\rho_{\sim} = - \rho_0 \frac{\partial z_{\sim}}{\partial z} = - \frac{\partial P_{\sim}}{\partial z}, \quad (15)$$

which means that

$$\frac{\rho}{\rho_0} = 1 - \frac{\partial z_{\sim}}{\partial z} = \frac{1}{1 + \partial z_{\sim} / \partial z_0}. \quad (16)$$

Since overtaking occurs when  $\partial z_{\sim} / \partial z_0 \leq -1$ , one notices that, as expected,  $\rho \rightarrow \infty$ , when it begins.



### 3. The linear (small signal) oscillations.

In order to facilitate our discussion of the non-linear oscillations we will in this section briefly discuss two characteristic, and naturally well known [2], linear solutions, viz. the "standing" plasma waves and the "travelling" plasma waves.

#### I. The "standing" plasma (or space charge) waves.

If we assume that an ideal, gridded velocity modulation gap is inserted at  $z_0 = 0$  (in the following we neglect  $z_0$  and thus write all solutions in terms of  $z$ ), that the streaming electrons have not been subject to perturbations of any kind before reaching this gap, and that the ac velocity produced by the same is  $(v_{\sim})_{z=0} = v_{\sim 0} \sin \omega t$ , we obtain

$$v_{\sim} = v_{\sim 0} \sin(\omega t - \alpha z) \cos(k_p z), \quad (17)$$

$$\begin{aligned} E_{\sim} &= \frac{m \omega}{e} v_{\sim 0} \sin(\omega t - \alpha z) \sin(k_p z) = \\ &= \frac{m v_0 v_{\sim 0}}{e} k_p \sin(\omega t - \alpha z) \sin(k_p z), \end{aligned} \quad (18)$$

which shows that  $\epsilon_0 E_{\sim}$  close to the gap grows like  $N_0 \propto v_{\sim 0}/v_0$ ,

$$i_{\sim} = - \frac{m \epsilon_0 \omega}{e} v_{\sim 0} \cos(\omega t - \alpha z) \sin(k_p z), \quad (19)$$

and

$$\begin{aligned} \frac{\rho_{\sim}}{\rho_0} &= \frac{v_{\sim 0}}{v_0} \left[ \frac{\omega}{\omega_p} \cos(\omega t - \alpha z) \sin(k_p z) - \right. \\ &\quad \left. - \sin(\omega t - \alpha z) \cos(k_p z) \right], \end{aligned} \quad (20)$$

where

$$\alpha = \omega/v_o. \quad (21)$$

We notice that the convection current is zero at the gap, but not so the differential space charge density. In the steady state, and relations (17) to (20) are steady state solutions, the velocity modulation produces an (normally small) ac space charge density in front of the gap. We furthermore notice from (20), that  $\rho_{\sim}/\rho_o$  becomes large for a very underdense stream, i.e. when  $\omega/\omega_p \ll 1$ . Overtaking thus rapidly occurs, when the stream becomes underdense.

If we have plasma resonance in the stream, i.e. when  $\omega = \omega_p$ , relations (17) and (20) yield

$$(v_{\sim})_{\omega=\omega_p} = \frac{1}{2} v_{\sim o} \left\{ \sin(\omega_p t) + \sin[\omega_p(t - 2z/v_o)] \right\}, \quad (22)$$

and

$$(\rho_{\sim}/\rho_o)_{\omega=\omega_p} = - \frac{v_{\sim o}}{v_o} \sin[\omega_p(t - 2z/v_o)]. \quad (23)$$

The ac velocity now has two characteristically different components, one a pure plasma oscillation independent of  $z$ , and the other a travelling "plasma wave" with the phase velocity  $v_o/2$ . The ac charge density only appears in the form of such a plasma (or space charge) wave. The pure plasma oscillation,  $\sin(\omega_p t)$ , as expected is not associated with any fluctuations in  $\rho$ . In this respect there is little difference between the stream and the stationary medium. For a more detailed discussion of the transition from a moving to a stationary medium, which is outside the scope of the present communication, the reader is referred to an earlier paper by Rydbeck [2].

It is of interest to note, that  $\rho_{\sim}$  now appears in the form of a pure travelling wave, a fact we will later make

use of to obtain an exact, and direct, solution of the density variations.

Finally, if we locate the modulation plane at such a position, that  $\cos(k_p z) = 0$ , it now appears from (20), that a density modulation,  $\rho_{\sim} = \rho_{\sim 0} \cos \omega t$ , at this plane (we tacitly assume that it is possible to perform such a modulation), produces a velocity modulation  $v_{\sim}/v_0 = (\rho_{\sim} \omega_p / \rho_0 \omega) \cos(k_p z) \sin(\omega t - \alpha z)$ ,  $k_p z \geq \pi/2$ . "Standing" plasma waves thus can be produced by pure velocity modulation, by pure density modulation or by a suitable combination of both. It should be noticed that the underdense beam ( $\omega \gg \omega_p$ ) is much more sensitive to velocity than to density modulation. By properly combining velocity and density modulation it is also possible to produce a "travelling" plasma (or space charge) wave.

## II. The "travelling" plasma (or space charge) wave.

If we produce the following velocity and density modulations at the modulation plane  $z = 0$ ,

$$\left( \frac{v_{\sim}}{v_0} \right)_{z=0} = \sin \omega t, \quad (24)$$

and

$$\left( \frac{\rho_{\sim}}{\rho_0} \right)_{z=0} = \mp \frac{v_{\sim 0}}{v_0} \frac{\omega}{\omega_p} \sin \omega t, \quad (25)$$

a "travelling" plasma wave is produced in the stream, viz.

$$\frac{v_{\sim}}{v_0} = \sin[\omega t - (\alpha \pm k_p) z], \quad (26)$$

$$\frac{\rho_{\sim}}{\rho_0} = \mp \frac{\alpha \pm k_p}{k_p} \sin[\omega t - (\alpha \pm k_p) z], \quad (27)$$

$$E_z = \pm \frac{m v_0 v_{\sim 0}}{e} k_p \cos[\omega t - (\alpha \pm k_p)z], \quad (28)$$

and

$$i_{\sim} = \pm \frac{m \epsilon_0 \omega_p \omega}{e} v_{\sim 0} \sin[\omega t - (\alpha \pm k_p)z]. \quad (29)$$

If the velocity and density modulation are  $180^\circ$  out of phase, a slow plasma wave,  $\sin[\omega t - (\alpha + k_p)z]$ , is produced, and when they are in phase, a fast plasma wave,  $\sin[\omega t - (\alpha - k_p)z]$ . It is important to note, that (for  $z > 0$ )  $\rho_{\sim}/\rho_0 = 0$ , when we have plasma resonance in the stream [ $\alpha = k_p$ ; compare also (22) and (23)]. The travelling plasma wave is of special interest in this connection, since its non-linear properties can be obtained exactly and direct, when overtaking does not occur, as will be shown in the following chapter 4.

4. The non-linear travelling plasma wave.

Since we are dealing with travelling waves, we now assume that all quantities can be expressed as functions of the same drift variable,

$$y = \omega t - \gamma z, \quad [\bar{E}_{\sim} = \bar{E}_{\sim}(y), \text{ etc.}] \quad (30)$$

where  $\gamma = \alpha + k_p$  for the slow plasma wave, and  $\gamma = \alpha - k_p$  for the fast plasma wave. It should be added, that it can be shown from our previous relations in chapter 2, that  $\gamma$  actually is a constant [the electrons according to (10) always oscillate harmonically with the angular frequency  $\omega_p$  around their undisturbed position]. The phase velocity of the travelling plasma wave thus becomes

$$v_{\text{phase}} = v_0 \frac{\alpha}{\alpha \pm k_p} = v_{ph_{\pm}} \quad (31)$$

By (30) we next obtain

$$(\omega - \gamma v) \frac{dv}{dy} = - \frac{q}{m} \bar{E}_{\sim}, \quad (\text{Eq. of motion}) \quad (32)$$

$$\rho_{\sim} = - \gamma \epsilon_0 \frac{d\bar{E}_{\sim}}{dy}, \quad (\text{Poisson's Eq.}) \quad (33)$$

and

$$(\omega - \gamma v) \frac{d\rho_{\sim}}{dy} = \gamma \rho \frac{dv}{dy}. \quad (\text{Eq. of continuity}) \quad (34)$$

If we introduce the "reduced" velocity wave function

$$w = \omega - \gamma v, \quad (35)$$

where it is to be noted that

$$w_0 = \omega - (\alpha \pm k_p) v_0 = \mp \omega_p, \quad (35a)$$

(the upper sign stands for the slow wave), relation (34) immediately yields

$$\frac{\rho}{\rho_0} = \frac{v_0}{v}, \quad (36)$$

i.e.

$$\frac{\rho_{\sim}}{\rho_0} = - \left( 1 - \frac{v_0}{v} \right), \quad (36a)$$

which can be written under the alternate form

$$\omega \rho_{\sim} = \gamma (v \rho - v_0 \rho_0). \quad (37)$$

The Eq. of motion, (32), by (35) can be written

$$\frac{1}{2} \frac{d}{dy} (v^2) = \frac{\gamma e}{m} E_{\sim}, \quad (38)$$

which yields

$$- \frac{e}{\epsilon_0 m} \rho_{\sim} = \frac{1}{2} \frac{d^2}{dy^2} (v^2), \quad (39)$$

or, finally by (36a), the non-linear wave equation

$$\frac{1}{2} \frac{d^2}{dy^2} (v^2) + \omega_p^2 \left( 1 - \frac{v_0}{v} \right) = 0. \quad (40)$$

The linearized (i.e. small signal) form of (40) becomes

$$\left( \frac{d^2}{dy^2} + 1 \right) v_{\sim} = 0, \quad (| \rho_{\sim} / \rho_0 | \ll 1). \quad (41)$$

which has the solution

$$v_{\sim} = A \sin(\omega t - \gamma z) + B \cos(\omega t - \gamma z),$$

i.e. travelling waves only.

The non-linear wave equation for  $E_{\sim}$  can easily be deduced from the previous relations. It becomes

$$\left[ \frac{d^2}{dy^2} + \left( \frac{\rho}{\rho_0} \right)^3 \right] E_{\sim} = \frac{d^2 E_{\sim}}{dy^2} + \left( 1 - \frac{\gamma \epsilon_0}{\rho_0} \frac{dE_{\sim}}{dy} \right)^3 E_{\sim} = 0. \quad (42)$$

Fortunately (40) can be solved exactly. Multiplication by  $dw/dy$  and integration twice yields

$$y - y_1 = \varphi + (1 - \rho_0/\rho_{\max}) \cos \varphi, \quad (43)$$

for the slow plasma wave, and

$$y - y_1 = -\varphi - (1 - \rho_0/\rho_{\max}) \cos \varphi, \quad (44)$$

for the fast plasma wave. Since  $y - y_1$  must everywhere be real,  $\varphi$  assumes the form

$$\varphi = \arcsin \left( \frac{1 - \rho_0/\rho}{1 - \rho_0/\rho_{\max}} \right) = \quad (45)$$

$$= \arcsin \left( \frac{\rho_0/\rho - 1}{\rho_0/\rho_{\min} - 1} \right) = \arcsin(\xi),$$

i.e.

$$\rho_0/\rho_{\min} = 2 - \rho_0/\rho_{\max}, \quad (46)$$

or

$$\frac{1}{2} \left( \frac{\rho_0}{\rho_{\min}} + \frac{\rho_0}{\rho_{\max}} \right) = 1. \quad (47)$$

Relations (43) and (44) can now be written

$$y - y_1 = \pm \left[ \arcsin(\xi) + \left( 1 - \frac{\rho_0}{\rho_{\max}} \right) \sqrt{1 - \xi^2} \right], \quad (43a)$$

from which  $y$  direct can be plotted as a function of  $\rho/\rho_0$ . One immediately infers from (43a), that

$$\pm (y - y_1) = 1 - \frac{\rho_0}{\rho_{\max}} (+ n 2\pi), \text{ when } \rho = \rho_0. \quad (48)$$

which demonstrates how the  $\rho_{\sim} = 0$  positions is displaced, when the stream is heavily modulated. Furthermore one notes, that

$$\pm (y - y_1) = \pi/2 (+ n 2\pi), \text{ when } \rho = \rho_{\max}, \quad (49)$$

and

$$\pm (y - y_1) = -\pi/2 (+n 2\pi), \text{ when } \rho = \rho_{\min}, \quad (50)$$

i.e. the distance between the space charge maximum and minimum is independent of  $\rho_0/\rho_{\max}$ , as it should be.

One easily proves from our previous relations, that

$$E_{\sim} = \pm \frac{m \omega_p^2}{e \gamma} \left(1 - \frac{\rho_0}{\rho_{\max}}\right) \sqrt{1 - \xi^2}, \quad (51)$$

and by (36) that

$$\begin{aligned} \frac{v_{\sim}}{v_0} &= \pm \left(\frac{\rho_0}{\rho_{\max}} - 1\right) \frac{\omega_p}{\omega \pm \omega_p} \xi = \\ &= \pm \left(\frac{\rho_0}{\rho} - 1\right) \frac{\omega_p}{\omega \pm \omega_p}. \end{aligned} \quad (52)$$

Relations (51) and (52) show that the positive and negative peak values of  $E_{\sim}$ , respectively  $v_{\sim}$  are equal even in the non-linear case.

When overtaking occurs  $\rho \rightarrow \infty$ , which thus takes place when

$$v_{\sim} \rightarrow v_{\sim \text{crit}} = \mp \frac{\omega_p}{\omega \pm \omega_p} v_0 = \mp \frac{\omega_p}{\omega} v_{ph_{\pm}}.$$

It thus appears that  $\frac{\omega}{\omega_p} v_{ph_{\pm}}/v_0$  can be regarded as a measure of the non-linearity. Furthermore one notices, as has already been pointed out in a previous section, that the critical ac velocity,  $v_{\sim \text{crit}}$ , is extremely small for a very underdense stream.



It is especially interesting to note, that for the resonant stream ( $\omega = \omega_p$ ),  $v_{crit}$  is equal to  $= v_0/2$  for the slow plasma wave and infinitely large for the fast plasma wave, which now transforms into a stationary plasma oscillation (see also chapter 3).

Furthermore it should be pointed out, that

$$v_{\sim crit} + v_0 = v_{ph_{\pm}}, \quad (\text{condition for overtaking}) \quad (53)$$

as might be expected.

Finally it should be mentioned that, according to (8), the displacement of the oscillating electron planes becomes

$$z_{\sim} = \pm \frac{1}{\gamma} \left(1 - \frac{\rho_0}{\rho_{\max}}\right) \sqrt{1 - \xi^2}. \quad (54)$$

The maximum displacements at overtaking thus are

$$\begin{aligned} z_{\sim} = z_{\sim crit} &= \pm \frac{1}{\gamma} = \pm \frac{\lambda_z}{2\pi} \frac{\omega}{\omega \pm \omega_p} = \\ &= \pm \frac{v_{ph_{\pm}}}{\omega} = \pm \frac{\lambda_{ph_{\pm}}}{2\pi}, \end{aligned} \quad (54a)$$

where  $\lambda_z = v_0 \lambda_0 / c_0$ , and  $\lambda_0$  is the vacuum wavelength.

It is worth noting from (52) that, even in the non-linear case,  $\rho_{\sim} = 0$  for the fast plasma wave, when  $\omega = \omega_p$ . The corresponding value of  $|z_{\sim crit}|$  then by (54a)  $\rightarrow \infty$ .

Finally one direct infers that, as it should be, (43a), (51) and (52) reduce to the linear expressions (27), (28), and (26), when  $\rho_{\sim max}/\rho_0 \ll 1$ .

In order to demonstrate the nature of the non-linear travelling plasma waves, we have in Figs. 1, 2 and 3 plotted  $\rho/\rho_0$ ,  $E_{\sim}/(E_{\sim})_{\rho=0}$ , and  $v_{\sim}/|v_{\sim max}|$  as functions of  $y = y_1$  for  $\rho_{\max}/\rho_0 = 1.1, 2.0, 10$  and  $100$ . If we, to take a typical example, assume that  $\omega/\omega_p = 5$  (a relatively underdense stream), these values, by (52), correspond to  $|v_{\sim max}|/v_0 = 0.0275$ ,

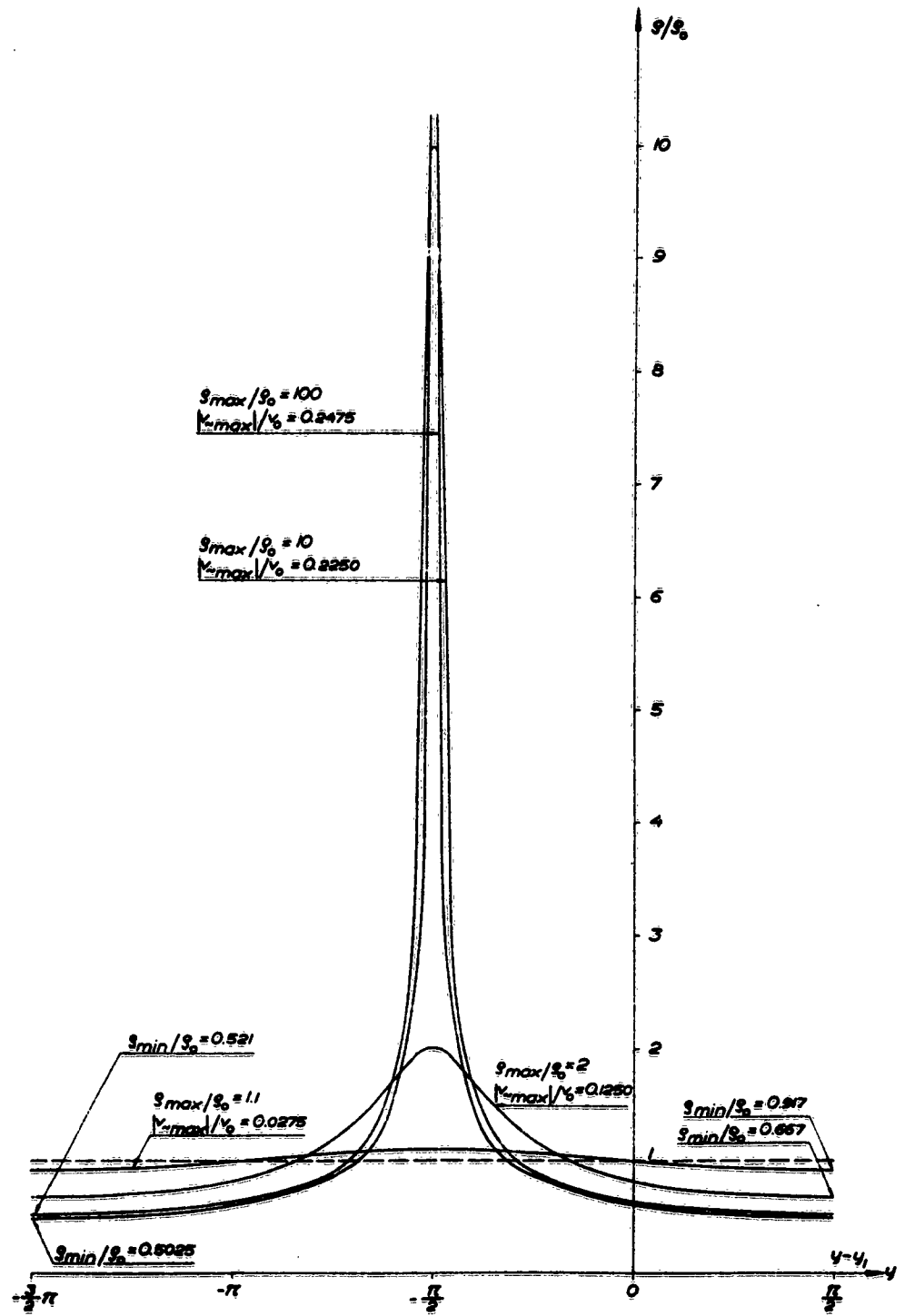


Fig. 1.  $\rho/\rho_0$  as function of  $y - y_1$  for the non-linear traveling plasma wave.

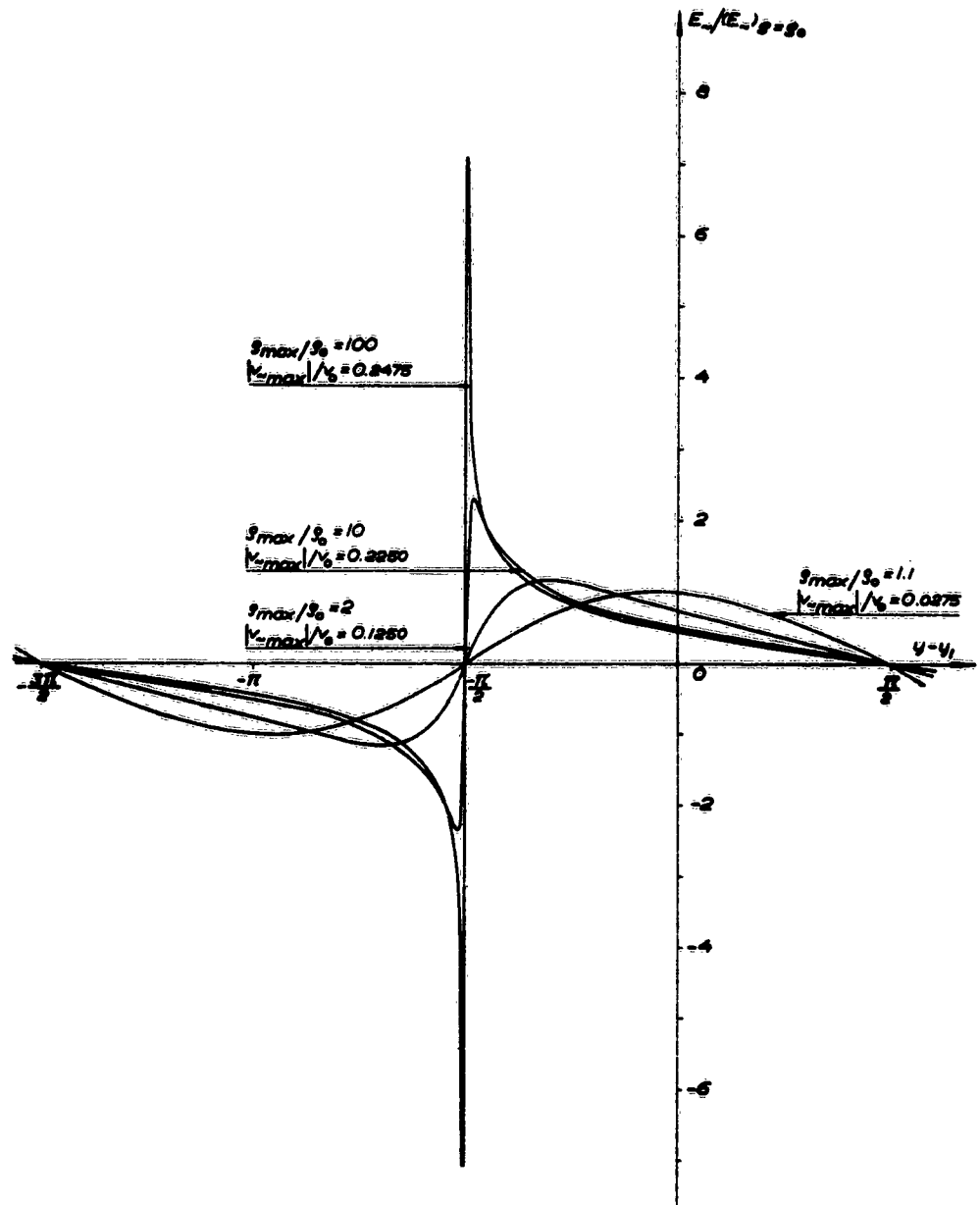


Fig. 2.  $E_{\sim}/(E_{\sim})_0 = g_0$  as function of  $y - y_1$  for the non-linear travelling plasma wave.

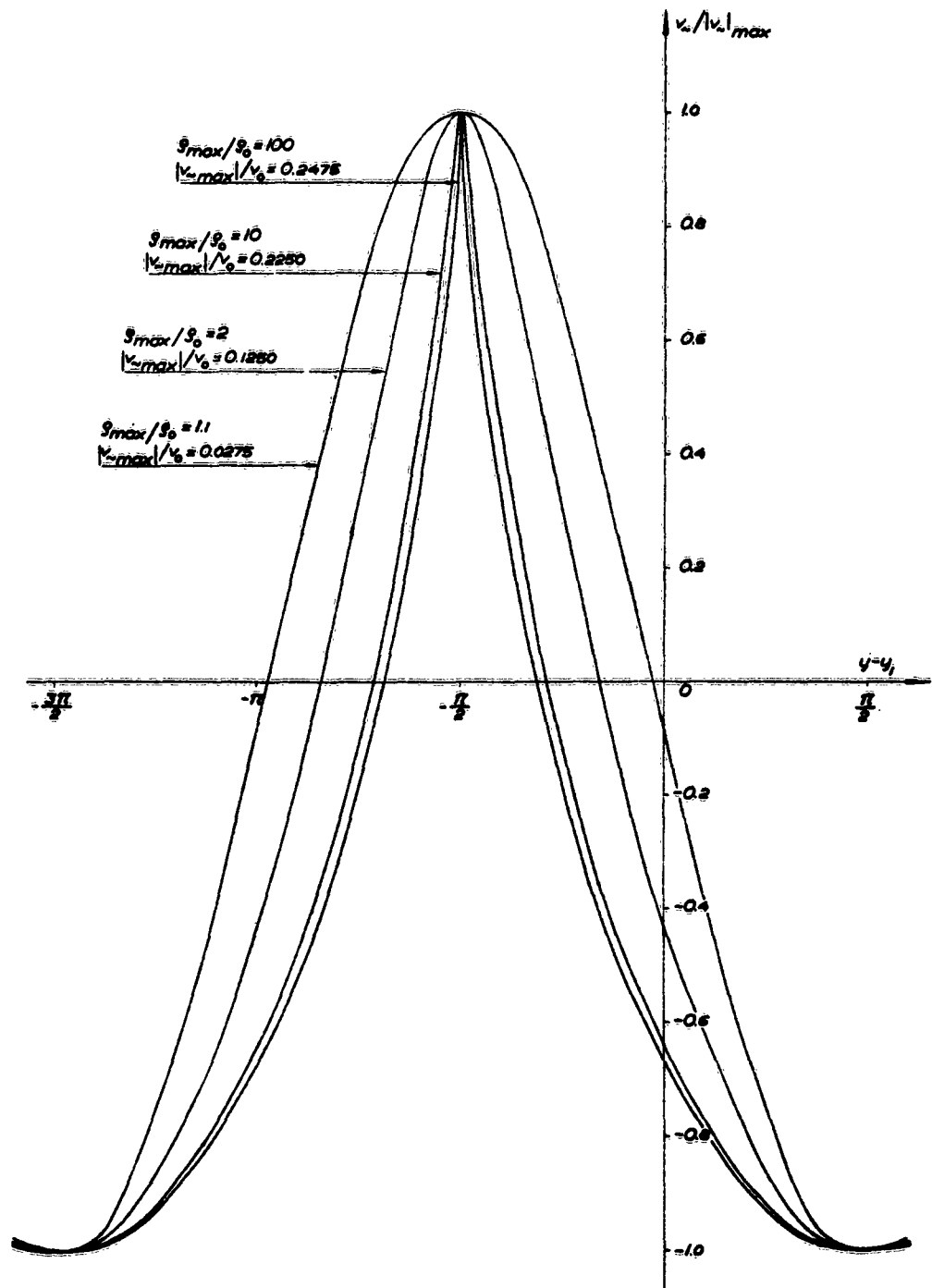


Fig. 3.  $v_e/|v_{e\max}|$  as function of  $y - y_1$  for the non-linear travelling plasma wave.

0.1250, 0.2250, and 0.2475 for the fast plasma wave ( $\rho_{\max}/\rho_0 = \infty$  would correspond to  $|v_{\max}|/v_0 = 0.2500$ , at which velocity overtaking begins). It appears from the three curves that the non-linear waves rapidly become very rich in higher harmonics. It is especially interesting to note the very intense charge bunching which becomes possible when  $v_{\sim}$  approaches its critical value (in the present numerical case  $v_0/4$ ). If such a stream runs through a multi-resonant system, for example the solar corona, an ionized medium in a plasma amplifier, or a series of cavities in a microwave device, very high harmonics may be excited. Whether a net harmonic radiation actually takes place or not from the interacting medium is a problem of its own, difficult and beyond the aims of the present report.

5. A comparison with Olving's method of non-linear analysis.

I. The non-linear travelling plasma wave.

Following Olving's approach [1], we now write down the electronic displacement for the fast plasma wave, viz.

$$z_{\sim} = - \frac{v_{\sim 0}}{v_0} \frac{1}{k_p} \cos[\omega t - (\alpha - k_p)z_0]. \quad (55)$$

By (16) this yields [see also (27)]

$$\frac{\rho}{\rho_0} = \frac{1}{1 - \frac{v_{\sim 0}}{v_0} \frac{\alpha - k_p}{k_p} \sin[\omega t - (\alpha - k_p)z_0]}, \quad (56)$$

where it has to be remembered, that  $z = z_0 + z_{\sim}$ . By these two relations  $\rho/\rho_0$  can successively be plotted as function of  $z$ , or of  $y = \omega t - (\alpha - k_p)z$  for that matter.

In this special case of a travelling plasma wave, one can, however, proceed in the following manner to obtain the same direct relations as in the previous chapter 4. If we introduce

$$\psi = \omega t - (\alpha - k_p)z_0 = \omega t - (\alpha - k_p)(z - z_{\sim}), \quad (57)$$

relation (56) can be written

$$\sin \varphi = \frac{1 - \rho_0/\rho}{1 - \rho_0/\rho_{\max}} = \xi, \quad (58)$$

i.e.

$$\varphi = \arcsin(\xi) (+ n 2\pi) \quad (59)$$

or

$$y = \omega t - (\alpha - k_p)z = \arcsin(\xi) - z_{\sim}(\alpha - k_p).$$

By (55) and (52) one obtains

$$z_{\sim}(\alpha - k_p) = (1 - \frac{\rho_0}{\rho_{\max}}) \sqrt{1 - \xi^2}, \quad (60)$$

which immediately yields

$$y = \arcsin(\xi) + (1 - \frac{\rho_0}{\rho_{\max}}) \sqrt{1 - \xi^2},$$

which is identical with our previous relation (43a).

When we have standing plasma waves in the system this simple transformation is not possible, except when  $\omega = \omega_p$ , as will be shown in the next section.

## II. The non-linear standing plasma wave.

Again following Olving we write

$$z_{\sim} = \frac{v_{\sim 0}}{v_0} \frac{1}{k_p} \sin(\omega t - \alpha z_0) \sin(k_p z_0). \quad (61)$$

By (16) this yields [see also (20)]

$$\frac{\rho}{\rho_0} = \frac{1}{1 - \frac{v_{\sim 0}}{v_0} \left[ \frac{\omega}{\omega_p} \cos(\omega t - \alpha z_0) \sin(k_p z_0) - \sin(\omega t - \alpha z_0) \cos(k_p z_0) \right]}, \quad (62)$$

where it again is to be remembered that  $z = z_0 + z_{\sim}$ . We furthermore have [see also (18) and (17)]

$$E_{\sim} = \frac{m v_0 v_{\sim 0}}{e} k_p \sin(\omega t - \alpha z_0) \sin(k_p z_0), \quad (63)$$

and

$$v_{\sim} = v_{\sim 0} \sin(\omega t - \alpha z_0) \cos(k_p z_0). \quad (64)$$

It appears from (62) and (61) that  $z_0$  and  $z_{\sim}$  can not

now be eliminated from (63), that can be written in the following form, viz.

$$\frac{\rho}{\rho_0} = \frac{1}{1 + \frac{v_0}{2v_p} \left\{ \frac{\omega + \omega_p}{\omega_p} \sin[\omega t - (\alpha + k_p) z_0] - \frac{\omega - \omega_p}{\omega_p} \sin[\omega t - (\alpha - k_p) z_0] \right\}} \quad (65)$$

If  $\omega = \omega_p$ , however,  $\rho/\rho_0$  becomes [see also (23)]

$$\frac{\rho}{\rho_0} = \frac{1}{1 + \frac{v_0}{v_p} \sin(\omega t - 2\alpha z_0)} \quad (66)$$

and we are left with one travelling wave only, since, as has been pointed out before, the stationary plasma oscillation does not produce any variations in  $\rho$ .

Following the same procedure as in section 1, we obtain

$$\omega t - 2\alpha z = \arcsin(\xi) - 2\alpha z_0. \quad (67)$$

Since  $z_0$  can be written

$$z_0 = \frac{v_0}{2k_p} \left\{ \cos[\omega t - (\alpha + k_p) z_0] - \cos[\omega t - (\alpha - k_p) z_0] \right\}, \quad (68)$$

we obtain, in the case of plasma resonance,

$$(z_0)_{\omega=\omega_p} = \frac{v_0}{2\alpha} [\cos(\omega t - 2\alpha z_0) - \cos \omega t], \quad (69)$$

i.e.,

$$(2\alpha z_0)_{\omega=\omega_p} = \left(1 - \frac{\rho_0}{\rho_{\max}}\right) (\sqrt{1 - \xi^2} - \cos \omega t). \quad (69a)$$

Relation (67) now yields the final result, viz.



$$y = \omega t - 2\alpha z =$$

$$= \arcsin(\xi) - \left(1 - \frac{\rho_0}{\rho_{\max}}\right) \sqrt{1 - \xi^2} + \left(1 - \frac{\rho_0}{\rho_{\max}}\right) \cos \omega t. \quad (70)$$

We notice, that for any given time,  $\rho/\rho_0$  now varies (we neglect the space phase difference) with  $z$  exactly as the non-linear pure travelling plasma wave. As a function of time, for a given position  $z$ , the variation is different however. This is due to the fact, that there is a pure plasma oscillation superimposed on the electronic displacement,  $z_{\sim}$ . The actual time variation, for fixed  $z$ , is easily obtained from Fig. 1, since one only has to correct  $y$  for the sinusoidal displacement  $(1 - \rho_0/\rho_{\max}) \cos \omega t$ . Methods of this type might be applicable to the theory of non-linear frequency mixing in streaming, ionized media.

\* \* \*

In order to compare the physical properties of the non-linear travelling and standing plasma waves, we have in Figs. 4, 5, 6, 7, 8, and 9 depicted the variation of  $\rho/\rho_0$ ,  $E_{\sim}/E_0$ , where  $E_0 = m v_0^2 k_p/e$ , and  $v_{\sim}/v_0$  as functions of  $z$  for  $\omega t = n 2\pi$ , and  $\omega t = n 2\pi + \pi/2$ . Furthermore, to demonstrate the importance and the effect of the electronic displacement,  $z_{\sim}$ , we have also, in the same Figs., plotted the former quantities (dashed curves) uncorrected for  $z_{\sim}$ , i.e. as functions of  $k_p z_0$ . It is interesting to note, that the displacement corrections tend to sharpen the peaks, when  $\omega t = n 2\pi$ , and broaden them when  $\omega t = n 2\pi + \pi/2$ . All curves have been

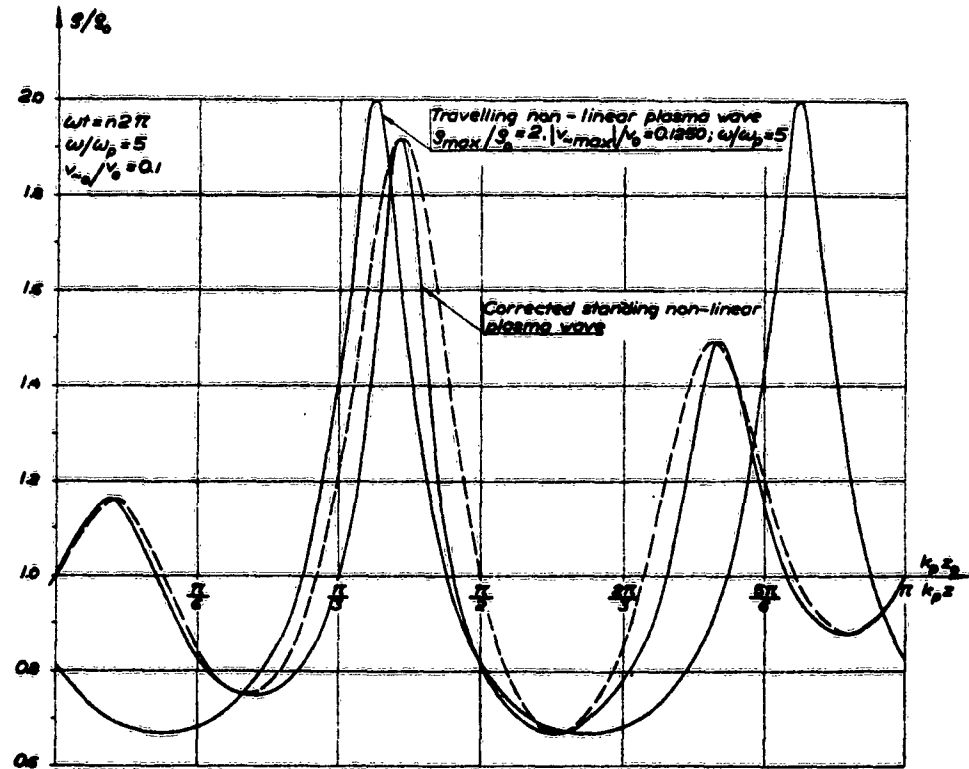


Fig. 4.  $\rho/\rho_0$  as function of  $x$  for non-linear travelling and standing plasma waves.

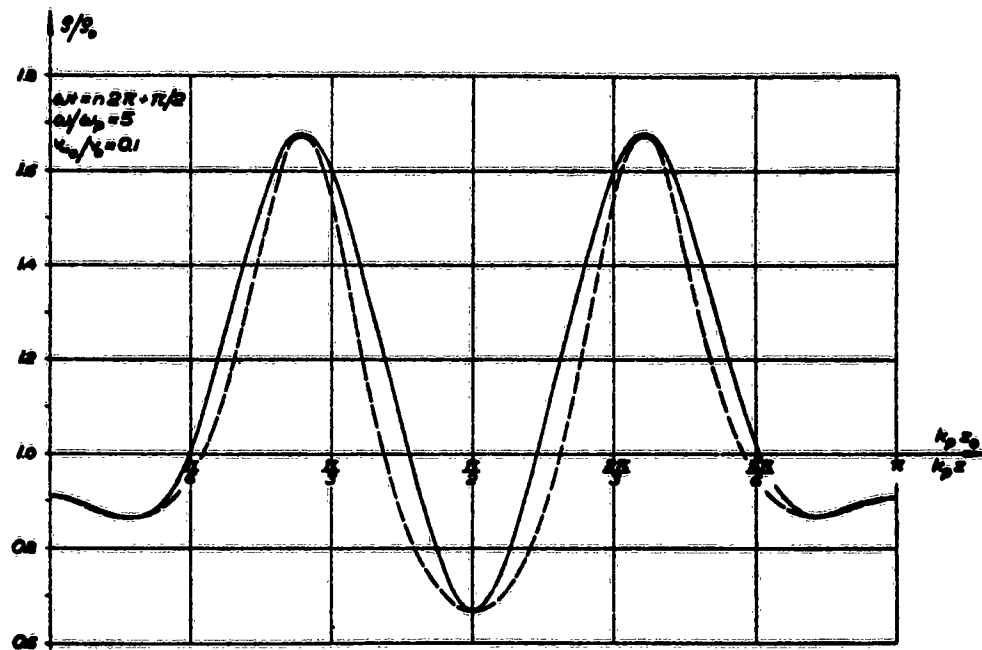


Fig. 5.  $\rho/\rho_0$  as function of  $z$  for a non-linear standing plasma wave ( $\omega t = n 2\pi + \pi/2$ ).

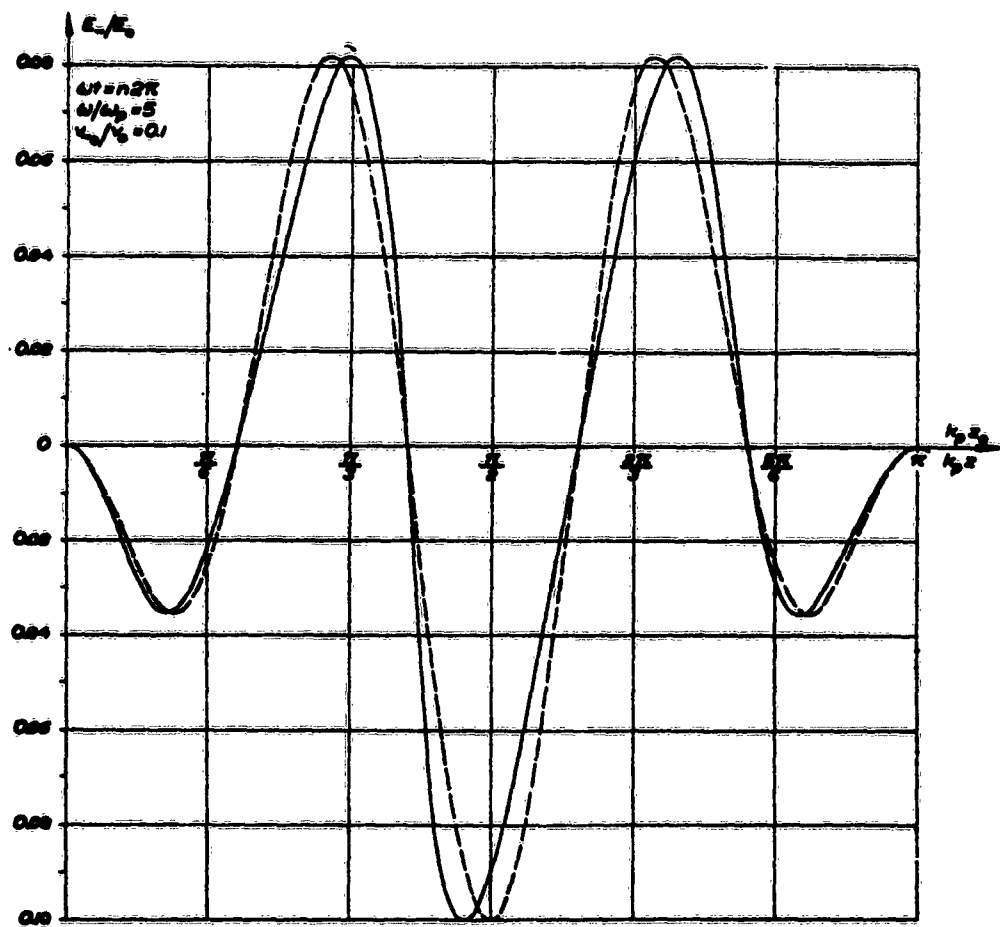


Fig. 6.  $E_x/E_0$  as function of  $x$  for the non-linear standing plasma wave ( $\omega t = n2\pi$ ).

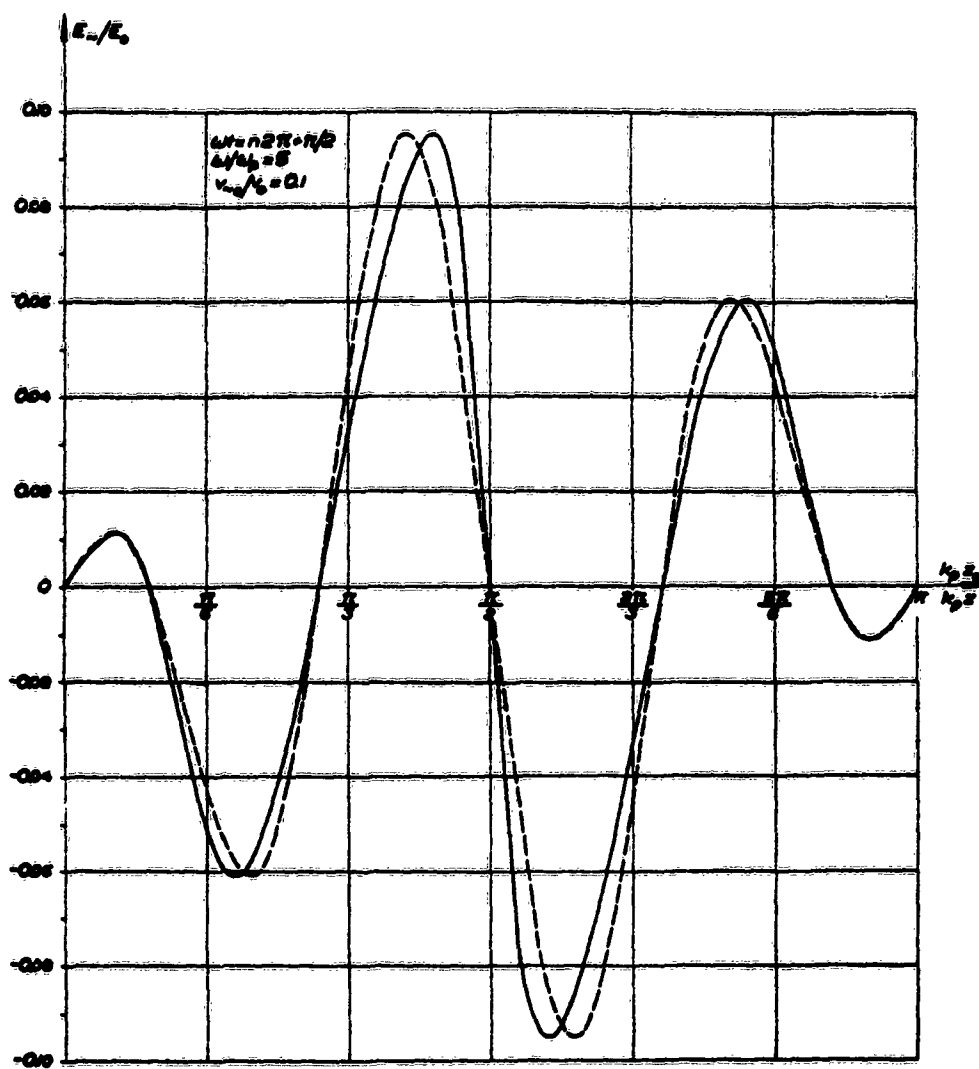


Fig. 7.  $E_z/E_0$  as function of  $z$  for the non-linear standing plasma wave ( $\omega t = n2\pi + \pi/2$ ).

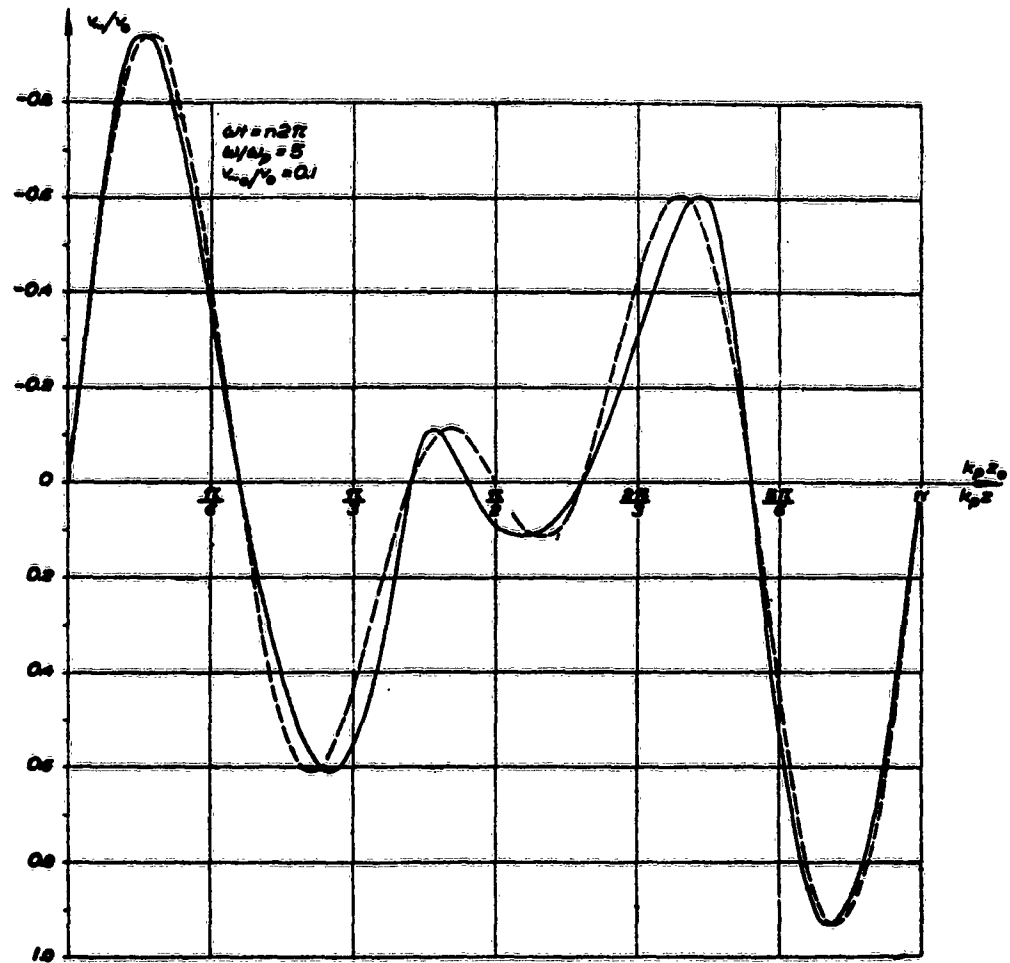


Fig. 8.  $v_z/v_0$  as function of  $z$  for the non-linear standing plasma wave ( $\omega t = n2\pi$ ).

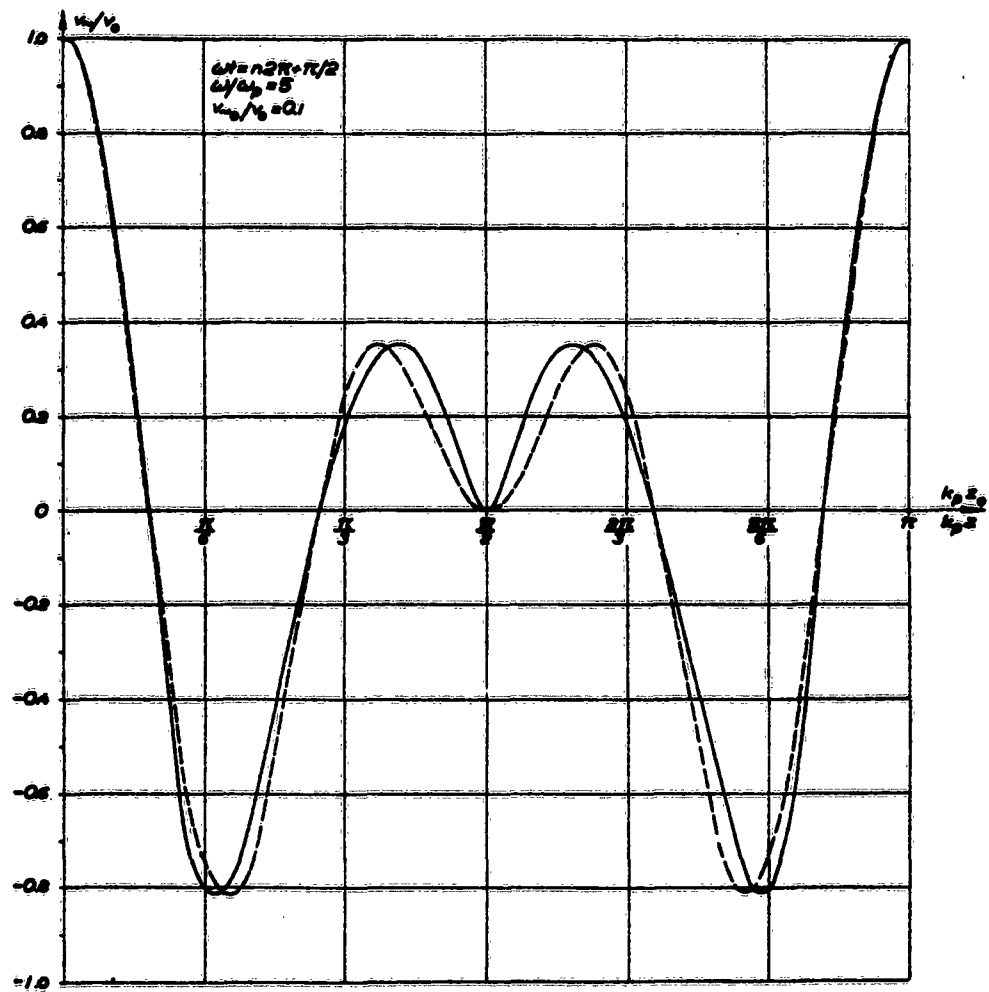


Fig. 9.  $v_z/v_0$  as function of  $z$  for the non-linear standing plasma wave ( $\omega t = n2\pi + \pi/2$ ).

plotted for  $v_{\sim 0}/v_0 = 0.10$  and  $\omega/\omega_p = 5$ , i.e. for a fairly underdense stream. This yields a maximum  $\rho/\rho_0$ -value of almost two.

Finally, in order to make a more direct comparison between the charge bunchings associated with the travelling and standing plasma waves, we have, in Fig. 4, replotted also the  $\rho/\rho_0$ -curve, for  $\rho_{\max}/\rho_0 = 2$  (and  $\omega/\omega_p = 5$ ), of Fig. 1. It is particularly interesting to note, that the peak bunching regions have almost the same shape for the two waves (which are modulated nearly equally much), in spite of their physically quite different nature.



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